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Convex Quadratic Programming Approach

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Abstract. The problem of determining a maximum matching or whether there exists a perfect matching, is very common in a large variety of applications and as been extensively studied in graph theory. In this paper we start to introduce a characterisation of a family of graphs for which its stability number is determined by convex quadratic programming. The main results connected with the recognition of this family of graphs are also introduced. It follows a necessary and sufficient condition which characterise a graph with a perfect matching and an algorithmic strategy, based on the determination of the stability number of line graphs, by convex quadratic programming, applied to the determination of a perfect matching. A numerical example for the recognition of graphs with a perfect matching is described. Finally, the above algorithmic strategy is extended to the determination of a maximum matching of an arbitrary graph and some related results are presented.

Key words: Convex programming, Graph theory, Maximum matching, Graph eigenvalues

1. Introduction

In this paper we deal with undirected simple graphs (that is, graphs where there is nor loops neither multiple edges), G, for which $V(G)$ denotes the set of nodes and $E(G)$ the set of edges. An element of $E(G)$, whose ends are the nodes i and j, is denoted by $\{i, j\}$. It is also assumed that G is of order $n \geqslant 1$ and size $m \geqslant 0$ (i.e., $|V(G)| = n \geqslant 1$ and $|E(G)| = m \geqslant 0$). The matrix A_G denotes the adjacency matrix of the graph G, that is, $A_G = (a_{ij})_{n\times n}$ is such that

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E(G) \\ 0 & \text{otherwise}. \end{cases}$$

Given $U \subset V(G)$, we denote by $G - U$ the subgraph induced by the set of nodes $V(G) \setminus U$ and by A_{G-U} its adjacency matrix. Given a node $i \in V(G)$, $N_G(i)$ is the neighbourhood of the node i, that is, $N_G(i) = \{j \in V(G) : \{i, j\} \in E(G)\}$.

A subset of nodes $S \subseteq V(G)$ is *stable* if no two nodes in S are linked by an edge. A stable set S is called a *maximum stable set* if there is no other stable set with greater number of nodes. The number of nodes in a maximum stable set of a graph G is called the *stability number* (or *independence number*) of G and it is denoted (as usually) $\alpha(G)$.

Given a subset of nodes of a graph G, $S \subseteq V(G)$, the vector $x \in \mathbb{R}^V$ with $x_v = 1$ if $v \in S$ and $x_v = 0$ if $v \notin S$ is called the characteristic vector of S.

Denoting by $\lambda_{\min}(A_G)$ the minimum eigenvalue of the adjacency matrix of a graph G, A_G, as it is well known, see (Cvetkovich, Doob and Sachs, 1979), if G has at least one edge, then $\lambda_{\min}(A_G) \leqslant -1$ and, $\lambda_{\min}(A_G) = -1$ if and only if every connected component of G is a complete subgraph. If $E(G) = \emptyset$ then $\lambda_{min}(A_G) = 0$.

As it is proved in Luz and Cardoso (1998), assuming that G has at least one edge, $\phi^*_{2,1}(G)$ is the best upper bound for $\alpha(G)$, among the optimal values of the family of convex quadratic programming problems

$$\phi^*_{a,b}(G) = \max\{a\hat{e}^T x - bx^T(\frac{1}{-\lambda_{min}(A_G)}A_G + I_n)x : x \geqslant 0\},$$

where $\hat{e}$ is the all-ones vector of $\mathbb{R}^n$, I_n is the identity matrix of order n, and a and b are real numbers such that $b \geqslant 0$ and $a - b \geqslant 1$. On the other hand, with the same assumption, according to Luz (1995), $\alpha(G) = \phi^*_{2,1}(G)$ if and only if for a maximum stable set S of G (and then for all),

$$-\lambda_{\min}(A_G) \leqslant \min\{|N_G(i) \cap S| : i \notin S\}. \tag{1.1}$$

Now, defining

$$(P_G^{\phi^*}) \;\; \phi^*(G) = \begin{cases} \phi^*_{2,1}(G) & \text{if } E(G) \neq \emptyset \\ \max\{2\hat{e}^T x - \| x \|^2, x \geqslant 0\} = n & \text{otherwise,} \end{cases}$$

as a direct consequence of the above results, we can conclude that for any graph, G, $\alpha(G) \leqslant \phi^*(G)$ and $\alpha(G) = \phi^*(G)$ if and only if the inequality (1.1) is fulfilled. Such upper bound, for $\alpha(G)$, also can be obtained from Motzkin–Straus result (Motzkin and Straus, 1965), which may be looked up in Gibbons et al (1997), from which we get

$$(\min\{x^T(A_G + I_n)x : x \geqslant 0, \hat{e}^T x = 1\})^{-1} = \alpha(G). \tag{1.2}$$

In fact, assuming $E(G) \neq \emptyset$ and then $\lambda_{\min}(A_G) \leqslant -1$, it follows that

$$x^T(\frac{1}{-\lambda_{\min}(A_G)}A_G + I_n)x \leqslant x^T(A_G + I_n)x \qquad \forall x \geqslant 0 \tag{1.3}$$

and hence, since by theorem 5 of Bomze (1998)

$$\begin{aligned} \phi^*(G) &= \max\{2\hat{e}^T x - x^T(\frac{1}{-\lambda_{\min}(A_G)}A_G + I_n)x : x \geqslant 0\} \\ &= (\min\{x^T(\frac{1}{-\lambda_{min}(A_G)}A_G + I_n)x : x \geqslant 0, \hat{e}^T x = 1\})^{-1}, \end{aligned}$$

applying (1.2) and (1.3) we obtain

$$\alpha(G) \leqslant \phi^*(G), \tag{1.4}$$

with equality if and only if $\lambda_{\min}(A_G) = -1$ or there is an optimal solution, x^*, for $(P_G^{\phi^*})$, such that $x^{*T} A_G x^* = 0$. It must be noted that if x^* is an optimal solution for $(P_G^{\phi^*})$ then $\hat{e}^T \frac{x^*}{\phi^*(G)} = 1$ and according to (1.2) and (1.4)

$$\frac{\phi^*(G)^2}{x^{*T}(A_G + I)x^*} \leqslant \alpha(G) \leqslant \phi^*(G). \tag{1.5}$$

The plan of the paper is as follows. In Section 2 a complete characterisation of the family of graphs with convex-QP stability number (where QP means quadratic programming) is provided as well the main results connected with its recognition. In Section 3 a necessary and sufficient condition that characterises graphs with a perfect matching is introduced as well an algorithmic strategy for the recognition of such graphs. In Section 4 a numerical example for the determination of a perfect matching of a graph, G, based on the determination of a maximum stable set of its line graph $L(G)$, by a convex quadratic programming approach is described. Finally, in Section 5, the recognition of a graph with a perfect matching is extended to the determination of a maximum matching of an arbitrary graph and some related results are presented.

2. Graphs with Convex-QP Stability Number

In Luz and Cardoso (1998), assuming that $G - \{v\}$ has at least one edge, it is proved that if $\phi_{2,1}^*(G-\{v\}) = \phi_{2,1}^*(G)$ and $\lambda_{\min}(A_{G-\{v\}}) \neq \lambda_{\min}(A_G)$ then $\alpha(G) = \phi_{2,1}^*(G)$ and, furthermore, if $\bar{x}^* \in \mathbb{R}^{n-1}$ is an optimal solution for $(P_{G-\{v\}}^{\phi^*})$, then x^* such that

$$x_i^* = \begin{cases} \bar{x}_i & \text{if } i \neq v \\ 0 & \text{otherwise,} \end{cases}$$

is the characteristic vector of a maximum stable set of G. Now we introduce the following generalisation.

THEOREM 1. *If $U \subseteq V(G)$ is such that $\phi^*(G-U) = \phi^*(G)$ and $\lambda_{min}(A_{G-U}) \neq \lambda_{min}(A_G)$ then $\phi^*(G) = \alpha(G)$ and, furthermore, any optimal solution of $(P_{G-U}^{\phi^*})$ define a characteristic vector of a maximum stable set of G.*

Proof. If $G - U$ has no edges then $\phi^*(G - U) = \alpha(G - U)$ and, according to the hypothesis,

$$\phi^*(G-U) = \alpha(G-U) \leqslant \alpha(G) \leqslant \phi^*(G) \Rightarrow \alpha(G) = \phi^*(G).$$

Furthermore $\alpha(G-U) = \alpha(G) = \phi^*(G-U)$ implies that the optimal solution of $(P_{G-U}^{\phi^*})$ is the characteristic vector of the maximum stable set $V(G) \setminus U$.

Let us suppose that $E(G-U) \neq \emptyset$, $\bar{x}^*$ is an optimal solution for $(P_{G-U}^{\phi^*})$ and x^*, is such that

$$x_j^* = \begin{cases} \bar{x}_j^*, & \text{if } j \in V(G) \setminus U \\ 0, & \text{if } j \in U. \end{cases}$$

Let $\bar{e}$ be a all ones vector with $p = n - |U|$ components. Then

$$\begin{aligned}\phi^*(G-U) &= 2\bar{e}^T\bar{x}^* - \bar{x}^{*T}(\frac{1}{-\lambda_{\min}(A_{G-U})}A_{G-U} + I_p)\bar{x}^* \\ &= 2e^Tx^* - x^{*T}(\frac{1}{-\lambda_{\min}(A_{G-U})}A_G + I_n)x^* \\ &\leqslant 2e^Tx^* - x^{*T}(\frac{1}{-\lambda_{min}(A_G)}A_G + I_n)x^* \\ &\leqslant \phi^*(G).\end{aligned} \tag{2.6}$$

Since $\phi^*(G-U) = \phi^*(G)$, then x^* is an optimal solution for $(P_G^{\phi^*})$ and

$$x^{*T}(\frac{1}{-\lambda_{min}(A_{G-U})}A_G + I_n)x^* = x^{*T}(\frac{1}{-\lambda_{min}(A_G)}A_G + I_n)x^*.$$

Thus $\lambda_{\min}(A_G) < \lambda_{\min}(A_{G-U})$ implies $x^{*T}A_Gx^* = 0$, that is, x^* is the characteristic vector of a maximum stable set of G, and then $\alpha(G) = \phi^*(G)$. □

Throughout this paper $\upsilon(G)$ stands for the optimal value of the convex quadratic programming problem

$$(P_G) \quad \upsilon(G) = \max\{2\hat{e}^Tx - x^T(H_G + I_n)x : x \geqslant 0\},$$

where

$$H_G = \begin{cases} \frac{1}{\lceil -\lambda_{\min}(A_G)\rceil}A_G & \text{if } \lambda_{\min}(A_G) \neq 0 \\ 0 & \text{otherwise,}\end{cases}$$

with $\lceil -\lambda_{\min}(A_G)\rceil$ denoting the least integer not less than the symmetric value of the minimum eigenvalue of A_G.

It must be noted that $\upsilon(G)$ remains an upper bound for $\alpha(G)$, which it is not less than $\phi^*(G)$, that is,

$$\alpha(G) \leqslant \phi^*(G) \leqslant \upsilon(G).$$

On the other hand, since the right hand side of the inequality (1.1) is always an integer, the lower bound of the necessary and sufficient condition for $\alpha(G) = \phi^*(G)$ and then for $\alpha(G) = \upsilon(G)^\star$ can be improved. So we can say that $\alpha(G) = \phi^*(G) = \upsilon(G)$ if and only if for a maximum stable set, S, of G (and then for all),

$$\lceil -\lambda_{\min}(A_G)\rceil \leqslant \min\{|N_G(i) \cap S| : i \notin S\}. \tag{2.7}$$

⋆ Note that if $\alpha(G) = \phi^*(G)$ then $(P_G^{\phi^*})$ has an integer $0-1$ optimal solution which, according to the Karush–Kuhn–Tucker optimality conditions, is also optimal for (P_G) and then $\phi^*(G) = \upsilon(G)$.

The main advantage of the use of $\upsilon(G)$ instead of $\phi^*(G)$ is that if $\lambda_{\min}(A_G)$ is not integer then the optimal solution of (P_G) is unique** and then is very easy to conclude if the equalities $\alpha(G) = \phi^*(G) = \upsilon(G)$ hold or not.

A graph G such that $\alpha(G) = \upsilon(G)$ is here in called a graph with *convex-QP stability number* and throughout this paper this class of graphs will be denoted by $\mathcal{Q}$.

According to the Karush–Kuhn–Tucker optimality conditions, $x^* \geqslant 0$ is an optimal solution for the convex quadratic programming problem (P_G) if and only if it is a solution of the linear complementarity problem

$$A_G x = \lceil -\lambda_{\min}(A_G) \rceil (\hat{e} - x) + y^*,$$

with $y^* \geqslant 0$ and $x^{*T} y^* = 0$.

As an immediate consequence, if x^* is an optimal solution for (P_G) then $x^{*T} A_G x^* = \lceil -\lambda_{\min}(A_G) \rceil x^{*T} (\hat{e} - x^*)$ and thus if $E(G) \neq \emptyset$ then

$$\upsilon(G) = 2\hat{e}^T x^* - x^{*T} (\frac{1}{\lceil -\lambda_{\min}(A_G) \rceil} A_G + I_n) x^* = \hat{e}^T x^* \geqslant 1.$$

Therefore, since any subset of nodes with characteristic vector x is a stable set iff $x^T A_G x = 0$, we conclude that $\alpha(G) = \upsilon(G)$ if and only if there is an optimal solution for (P_G), x^*, such that $x^* \in \{0, 1\}^n$. The same conclusion may be obtained adapting the inequalities (1.5), assuming that x^* is an optimal solution for (P_G) and replacing $\phi^*(G)$ by $\upsilon(G)$. As a direct consequence of the next theorem, it is also easy to conclude that if x^* is an optimal solution for (P_G) then $\forall i \in V(G)\ \ 0 \leqslant x_i^* \leqslant 1$.

THEOREM 2. *Let a_G^i be the i-th row of the matrix A_G. Then the n-tuple of real numbers x^* is an optimal solution for (P_G) if and only if $\forall i \in V(G)\ \ x_i^* = \max\{0, 1 - r_i^*\}$, where $r_i^* = \frac{a_G^i x^*}{\lceil -\lambda_{min}(A_G) \rceil}$.*

Proof. Let x^* be an optimal solution for (P_G), and let $\bar{e}$, $\bar{x}^*$ and $\bar{x}$ the subvectors of $\hat{e}$, x^* and x, respectively, without the i-th component. Let

$$f_{G-\{i\}}(\bar{x}) = 2\bar{e}^T \bar{x} - \bar{x}^T (\frac{1}{\lceil -\lambda_{\min}(A_G) \rceil} A_{G-\{i\}} + I_{n-1}) \bar{x}.$$

First, it must be observed that

$$\begin{aligned} \upsilon(G) &= \max\{f_{G-\{i\}}(\bar{x}) + 2x_i - x_i^2 - 2x_i \frac{a_G^i x}{\lceil -\lambda_{min}(A_G) \rceil} : \bar{x}, x_i \geqslant 0\} \\ &= f_{G-\{i\}}(\bar{x}^*) + 2x_i^* - x_i^{*2} - 2x_i^* r_i^* \\ &= f_{G-\{i\}}(\bar{x}^*) + \psi(x_i^*) \\ &= f_{G-\{i\}}(\bar{x}^*) + \max\{\psi(x_i) : x_i \geqslant 0\}, \end{aligned}$$

** It must be noted that if $-\lambda_{\min}(A_G) < \lceil -\lambda_{\min}(A_G) \rceil$ then the objective function of (P_G) is strictly convex.

where $\psi(x_i) = 2x_i - x_i^2 - 2x_i r_i^*$.

Since $\frac{d}{dx_i}\psi(x_i) = 2(1 - r_i^*) - 2x_i$ and $\frac{d^2}{dx_i^2}\psi(x_i) = -2$, we can conclude that $\psi(x_i)$ is a strictly concave function which attains its maximum on a critical point, i.e., on the solution of equation

$$\frac{d}{dx_i}\psi(x_i) = 0 \Leftrightarrow x_i = 1 - r_i^*.$$

Therefore, since $\psi(x_i^*) = \max\{\psi(x_i) : x_i \geqslant 0\}$, we have

$$x_i^* = \max\{0, 1 - r_i^*\}.$$

Conversely, let us suppose that there is an n-tuple of real numbers, x^*, such that $\forall i \in V(G) \;\; x_i^* = \max\{0, 1 - r_i^*\}$.

Thus $\forall i \in V(G) \;\; x_i^* = 1 - r_i^* + y_i$, with

$$y_i = \begin{cases} -(1 - r_i^*) & \text{if } (1 - r_i^*) < 0 \\ 0 & \text{otherwise,} \end{cases}$$

and then $y_i \geqslant 0$. So $\forall i \in V(G)$

$$r_i^* = 1 - x_i^* + y_i \Leftrightarrow a_G^i x^* = \lceil -\lambda_{min}(A_G)\rceil (1 - x_i^* + y_i).$$

Therefore, setting $y_i^* = \lceil -\lambda_{min}(A_G)\rceil y_i \;\; \forall i \in V(G)$, we get the system of equations

$$\begin{aligned} a_G^i x^* &= \lceil -\lambda_{\min}(A_G)\rceil (1 - x_i^*) + y_i^*, \quad i \in V(G) \\ A_G x^* &= \lceil -\lambda_{min}(A_G)\rceil (\hat{e} - x^*) + y^*, \end{aligned}$$

which is equivalent to the Karush–Kuhn–Tucker optimality conditions (since $x^{*T} y^* = 0$). □

It is an obvious conclusion that if a graph, G, has no edges then, the optimal solution of (P_G) is the characteristic vector of the maximum stable set induced by $V(G)$, and thus $\alpha(G) = \upsilon(G) = |V(G)|$.

The class of graphs with convex-QP stability number is not hereditary (Lozin and Cardoso, 1999). Though, according to the next theorem, this class of graphs is closed under deletion of α–reducible sets of nodes (defining an α–reducible set of nodes as being a subset $U \subset V(G)$ such that $\alpha(G) = \alpha(G - U)$).

Before to proceed it must be noted that, by interlacing properties, if $U \subseteq V(G)$ then $\lambda_{\min}(A_G) \leqslant \lambda_{\min}(A_{G-U})$ and therefore, by (2.6),

$$\upsilon(G - U) \leqslant \upsilon(G). \tag{2.8}$$

THEOREM 3. *If $G \in \mathcal{Q}$ and $U \subseteq V(G)$ is such that $\alpha(G) = \alpha(G - U)$ then $G - U \in \mathcal{Q}$.*

Proof. Since U is such that $\alpha(G) = \alpha(G - U)$ therefore, from the inequalities

$$\alpha(G - U) \leqslant \upsilon(G - U) \leqslant \upsilon(G),$$

we can conclude that $\alpha(G) = \upsilon(G)$ implies $\alpha(G - U) = \upsilon(G - U)$. □

As a consequence of the above theorem, if $G \in \mathcal{Q}$ then there is a set $U \subseteq V(G)$ such that $|U| = |V(G)| - \upsilon(G)$ and $\forall T \subseteq U, \quad G - T \in \mathcal{Q}$. Note that if S is a maximum stable set of G then

$$\forall T \subseteq V(G) \setminus S,\ \alpha(G) = \alpha(G - T) \leqslant \upsilon(G - T) \leqslant \upsilon(G),$$

and therefore $\alpha(G) = \upsilon(G) \Rightarrow \alpha(G - T) = \upsilon(G - T) = \upsilon(G)$.

The following results provide an algorithmic strategy for the recognition of graphs with quadratic stability number.

THEOREM 4. *If there exists $v \in V(G)$ such that*

$$\upsilon(G) \neq \max\{\upsilon(G - \{v\}), \upsilon(G - N_G(v))\}$$

then $G \notin \mathcal{Q}$.

Proof. Since by (2.8) $\upsilon(G) \geqslant \upsilon(G - U)$ $\forall U \subseteq V(G)$, the hypothesis of theorem implies that $\upsilon(G) > \max\{\upsilon(G - \{v\}), \upsilon(G - N_G(v))\}$.

Let S be a maximum stable set of G. If $v \notin S$ then

$$\alpha(G) = \alpha(G - \{v\}) \leqslant \upsilon(G - \{v\}) < \upsilon(G).$$

If $v \in S$ then $\alpha(G) = \alpha(G - N_G(v)) \leqslant \upsilon(G - N_G(v)) < \upsilon(G)$. □

As immediate consequence of the above theorem, if $G \in \mathcal{Q}$ then

$$\forall v \in V(G),\ \upsilon(G) = \max\{\upsilon(G - \{v\}), \upsilon(G - N_G(v))\}.$$

THEOREM 5. *Consider that $\upsilon(G) = \max\{\upsilon(G - \{v\}), \upsilon(G - N_G(v))\}$ and that $\upsilon(G - \{v\}) \neq \upsilon(G - N_G(v))$.*

1. *If $\upsilon(G) = \upsilon(G - \{v\})$ then $G \in \mathcal{Q} \Leftrightarrow G - \{v\} \in \mathcal{Q}$.*
2. *If $\upsilon(G) = \upsilon(G - N_G(v))$ then $G \in \mathcal{Q} \Leftrightarrow G - N_G(v) \in \mathcal{Q}$.*

Proof.

1. Let us suppose that $G \in \mathcal{Q}$. Since $\alpha(G) = \upsilon(G) > \upsilon(G - N_G(v)) \geqslant \alpha(G - N_G(v))$, we can conclude that $\alpha(G) > \alpha(G - N_G(v))$. Thus, if S is a maximum stable set for G, then $N_G(v) \cap S \neq \emptyset$ and therefore $v \notin S$. So

   $$\alpha(G - \{v\}) = \alpha(G) = \upsilon(G) = \upsilon(G - \{v\}) \Rightarrow G - \{v\} \in \mathcal{Q}.$$

   Conversely, supposing that $G - \{v\} \in \mathcal{Q}$, according to the inequalities

   $$\alpha(G - \{v\}) \leqslant \alpha(G) \leqslant \upsilon(G) = \upsilon(G - \{v\}),$$

   we can conclude that $\alpha(G) = \upsilon(G)$.

2. Let us suppose that $G \in \mathcal{Q}$. Then, according to the hypothesis,

$$\alpha(G) = \upsilon(G) > \upsilon(G - \{v\}) \geqslant \alpha(G - \{v\}),$$

and then $\alpha(G - \{v\}) < \alpha(G)$. Therefore, if S is a maximum stable set then $v \in S,\ N_G(v) \cap S = \emptyset$ and $\alpha(G - N_G(v)) = \alpha(G)$.
Thus $\alpha(G) = \alpha(G - N_G(v)) \leqslant \upsilon(G - N_G(v)) \leqslant \upsilon(G)$ and the assumption that $G \in \mathcal{Q}$ implies $\alpha(G - N_G(v)) = \upsilon(G - N_G(v))$.
Conversely, supposing $G - N_G(v) \in \mathcal{Q}$, according to the hypothesis we know that $\alpha(G - N_G(v)) \leqslant \alpha(G) \leqslant \upsilon(G) = \upsilon(G - N_G(v))$ and then $\alpha(G) = \upsilon(G)$.
□

None of the theorems 4 and 5 can be applied when all of the following equalities hold

$$\forall v \in V(G) \qquad \upsilon(G) = \upsilon(G - \{v\}) = \upsilon(G - N_G(v)).$$

However the next theorem provides a branching strategy for such graphs (or subgraphs).

THEOREM 6. *If there exists $v \in V(G)$ such that $\upsilon(G) = \upsilon(G - \{v\}) = \upsilon(G - N_G(v))$ then*

$$G \in \mathcal{Q} \text{ if and only if } \begin{cases} G - N_G(v) \in \mathcal{Q} \\ \quad\text{or} \\ G - \{v\} \in \mathcal{Q} \end{cases}$$

Proof. Let us suppose that $G \in \mathcal{Q}$ and S is a maximum stable set of G. If $v \in S$ then $\alpha(G) = \alpha(G - N_G(v)) \leqslant \upsilon(G - N_G(v)) = \upsilon(G) \Rightarrow \alpha(G - N_G(v)) = \upsilon(G - N_G(v))$. If $v \notin S$ then $\alpha(G) = \alpha(G - \{v\}) \leqslant \upsilon(G - \{v\}) = \upsilon(G) \Rightarrow \alpha(G - \{v\}) = \upsilon(G - \{v\})$.

Conversely let us suppose that $G - U \in \mathcal{Q}$, with $U = \{v\}$ or $U = N_G(v)$. Then, since $\alpha(G - U) \leqslant \alpha(G) \leqslant \upsilon(G) = \upsilon(G - U)$, we can conclude that $\alpha(G) = \upsilon(G)$. □

There are a large variety of graphs with convex-QP stability number. For instance, as will be proved in Section 5, if G is a connected graph with an even number of edges, then $L(L(G)) \in \mathcal{Q}$, where $L(G)$ denotes the line graph of G.

The *line graph* of a graph G it is constructed by taking the edges of G as nodes of $L(G)$, and joining two nodes in $L(G)$ whenever the corresponding edges in G have a common node.

3. Characterisation of Graphs with a Perfect Matching

Given the graph G, a *matching* in G is a subset of edges, $M \subseteq E(G)$, no two of which have a common node. A matching with maximum cardinality is designated *maximum matching*. On the other hand if for each node $v \in V(G)$ there is one edge of the matching M incident with v, then M is called a *perfect matching*.

The problem of determining a maximum matching or whether there exists a perfect matching is very common in a large variety of applications and as been extensively studied in graph theory. There are several very readable texts about matching theory, among which we can refer, for instance, the classical monograph of Lovász and Plummer (Lovász and Plummer, 1986) or the survey of Pulleyblank (Pulleyblank, 1995). The determination of a maximum stable set of a line graph $L(G)$ is equivalent to the determination of a maximum matching of G. Therefore, since $\alpha(L(G)) \leqslant \upsilon(L(G))$, the optimal solution of $(P_{L(G)})$ is an upper bound on the number of elements of a maximum matching of G. Based on the Edmonds perfect matching algorithm, introduced in his landmark paper (Edmonds, 1965), polynomial-time algorithms have been developed for the determination of a maximum matching of a graph G.

A basic property of line graphs is that they are claw-free (that is, they are graphs which contains no induced subgraph isomorph to $K_{1,3}$). In Minty (1980) and Sbihi (1980) polynomial-time algorithms for the determination of maximum stable sets of claw-free graphs were introduced. However, none of them utilize a convex quadratic programming approach.

Let us denote by $B_G = (b_{ve})_{n\times m}$ (where $n = |V(G)|$ and $m = |E(G)|$) the node edge incident matrix of a graph G, that is, such that

$$b_{ve} = \begin{cases} 1 & \text{if the node } v \text{ is incident with the edge } e \\ 0 & \text{otherwise.} \end{cases}$$

Then $B_G^T B_G = A_{L(G)} + 2I_m$ is a positive semidefinite matrix and

$$\forall u \in \mathbb{R}^m \setminus \{0\}, u^T B_G^T B_G u = u^T A_{L(G)} u + 2||u||^2 \geqslant 0 \Rightarrow \frac{u^T A_{L(G)} u}{||u||^2} \geqslant -2.$$

So, the minimum eigenvalue of $A_{L(G)}$ is not less than -2 and if the Kernel of B_G, $Ker(B_G) = \{u \in \mathbb{R}^n : B_G u = 0\}$, is nontrivial and $u \in Ker(B_G) \setminus \{0\}$, then $A_{L(G)} u = B_G^T B_G u - 2u = -2u$. As a consequence, -2 is an eigenvalue of $A_{L(G)}$ and then $\lambda_{\min}(A_{L(G)}) = -2$.

For a connected graph, G, $\lambda_{\min}(A_{L(G)}) = -2$ if and only if G has an even cycle or two odd cycles (Doob, 1973). On the other hand, given a connected graph G with n nodes and m edges, such that $\lambda_{\min}(A_{L(G)}) = -2$, then the multiplicity of this eigenvalue is $m - n + 1$ if G is bipartite, and $m - n$ otherwise (Doob, 1973).

Now, before to introduce the main result of this section, let us remind that a graph has a perfect matching if and only if each of its components has a perfect matching. Hence, in order to get a characterisation of graphs with perfect matchings it suffices to consider the case that G is connected.

THEOREM 7. *A connected graph G of order $n > 1$, such that $L(G)$ is not complete, has a perfect matching if and only if $L(G) \in \mathcal{Q}$.*

Proof. Since $L(G)$ is a line graph which is connected (since G is connected) and not complete, then

$$-2 \leqslant \lambda_{\min}(A_{L(G)}) < -1 \Leftrightarrow 1 < -\lambda_{min}(A_{L(G)}) \leqslant 2.$$

Let us suppose that $S(G)$ is a perfect matching of G for which $S(L(G))$ is the corresponding independent set of nodes of $L(G)$. Let $e \in E(G) \setminus S(G)$ be the edge which corresponds in $L(G)$ to the node v_e. Since $S(G)$ is a perfect matching of G, if $e = \{i, j\}$ then there are two edges $e_i, e_j \in S(G)$ such that the node i is incident with the edge e_i and the node j is incident with the edge e_j. Thus, by construction, the node v_e is adjacent to the nodes $v_{e_i}, v_{e_j} \in S(L(G))$ and we can conclude that

$$\forall v_e \notin S(L(G))\ |N_{L(G)}(v_e) \cap S(L(G))| \geqslant 2.$$

Furthermore, since $L(G)$ is a claw-free graph we can conclude that $|N_{L(G)}(v_e) \cap S(L(G))| = 2$. Therefore

$$\lceil -\lambda_{\min}(A_{L(G)}) \rceil = 2 \leqslant \min\{|N_{L(G)}(v_e) \cap S(L(G))|\ :\ v_e \notin S(L(G))\}$$

and, according to (2.7), $\alpha(L(G)) = \upsilon(L(G))$.

Conversely let us suppose that there is no perfect matching for G and that $S(G)$ is a maximum matching. Then there is a node $k \in V(G)$ such that k is not incident with any edge of $S(G)$. On the other hand, since G has no isolated nodes, the node k is incident with an edge $\tilde{e} = \{k, j\} \in E(G)$, and there is an edge $\ddot{e} \in S(G)$, such that j is incident with $\ddot{e}$ (otherwise $S(G)$ would be not a maximum matching). Thus, the node of $L(G)$ which corresponds to $\tilde{e}$, $v_{\tilde{e}}$, does not belong to the maximum stable set $S(L(G))$, and, by construction of $L(G)$, $v_{\ddot{e}}$ is the only node of $S(L(G))$ adjacent to $v_{\tilde{e}}$. Therefore,

$$\min\{|N_{L(G)}(v_e) \cap S(L(G))| :\ v_e \notin S(L(G))\} = 1 < 2 = \lceil -\lambda_{\min}(L(G)) \rceil$$

and, once again, by (2.7), $\alpha(L(G)) \neq \upsilon(L(G))$. □

It must be noted that if $|E(G)| > 1$ then only the triangles and the stars are graphs G for which $L(G)$ is a complete graph. For these graphs, however, it is very easy to find a maximum matching.

The next theorem provides an easy way to find optimal solutions for $(P_{L(G)})$, when G has a perfect matching.

THEOREM 8. *If G is connected and $L(G) \in \mathcal{Q}$ then the optimal solutions of $(P_{L(G)})$ are critical points for its objective function,*

$$f_{L(G)}(x) = 2\hat{e}^T x - x^T(H_{L(G)} + I_m)x.$$

Proof. If $|E(G)| = 1$ the proof is trivial. Let us suppose that $|E(G)| > 1$ and let S be a maximum stable set for $L(G)$. Since $L(G) \in \mathcal{Q}$, then by (2.7) $\forall v \notin S\ \ |N_{L(G)}(v) \cap S| = 2$ and, the characteristic vector of S, x^*, is an optimal solution for $(P_{L(G)})$. Therefore, by the Karush–Kuhn–Tucker optimality conditions, $\exists y^* \geqslant 0$ such that

$$A_{L(G)}x^* = 2(\hat{e} - x^*) + y^* \quad \wedge \quad x^{*T}y^* = 0,$$

where $\hat{e}$ denotes a all ones vector of $\mathbb{R}^m$, with $m = |E(G)| = |V(L(G))|$. Denoting by $a^i_{L(G)}$ the i-th row of $A_{L(G)}$, we have the following equalities:

$$\forall i \in S\ a^i_{L(G)}x^* = \sum_{j \in N_{L(G)}(i)} x^*_j = 0 = 2(1 - x^*_i) + y^*_i \Rightarrow y^*_i = 0,$$
$$\forall i \notin S\ a^i_{L(G)}x^* = \sum_{j \in N_{L(G)}(i)} x^*_j = 2 = 2(1 - x^*_i) + y^*_i \Rightarrow y^*_i = 0.$$

Then $y^* = 0$ and

$$A_{L(G)}x^* = \lceil -\lambda_{min}(A_{L(G)})\rceil(\hat{e} - x^*) \Leftrightarrow \nabla f_{L(G)}(x^*) = 0,$$

where $\nabla f_{L(G)}(x)$ denotes the gradient of the objective function of $(P_{L(G)})$. □

If for a connected graph G of order $n > 1$, such that $L(G)$ is not complete, there is a critical point, $x^* \in \{0,1\}^n$, of the objective function of $(P_{L(G)})$, $f_{L(G)}(x) = 2\hat{e}^T x - x^T(H_{L(G)} + I_m)x$, then it is obvious that $L(G) \in \mathcal{Q}$. Therefore, to recognise if $L(G) \in \mathcal{Q}$ is equivalent to recognise if the system $A_{L(G)}x = \lceil -\lambda_{min}(A_{L(G)})\rceil(\hat{e} - x)$ has a $0-1$ solution. However, as it is well known, in general, the problem o finding a $0-1$ solution among the ones of a system of linear equations is an hard problem.

Since, if a connected graph, G, has a perfect matching then $\upsilon(L(G)) = \frac{|V(G)|}{2} \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, let $\mathcal{G}$ be the set of connected graphs G of order $n > 1$, such that $\upsilon(L(G)) = \frac{n}{2} \in \mathbb{Z}$ (and then they are neither triangles, K_3, nor stars, $K_{1,p}$, with which $\upsilon(L(K_3)) = \upsilon(L(K_{1,p})) = 1$). Thus we have the following algorithmic strategy for the recognition of graphs $G \in \mathcal{G}$ such that $L(G) \in \mathcal{Q}$.

0. *Algorithm* (to recognise graphs $G \in \mathcal{G}$ such that $L(G) \in \mathcal{Q}$)

1. Set $W = V(L(G))$;

2. Let $x^*_{L(G)}$ be an optimal solution for $(P_{L(G)})$.

2.1 If $x^* \in \{0,1\}^{|V(L(G))|}$ then STOP ($L(G) \in \mathcal{Q}$);

3. If $\lambda_{\min}(A_{L(G)}) \notin \mathbb{Z}$ then STOP ($L(G) \notin \mathcal{Q}$);

4. Choose $w \in W$ and set $W = W \setminus \{w\}$;

5. If $\upsilon(L(G)) \notin \{\upsilon(L(G) - \{w\}), \upsilon(L(G) - N_{L(G)}(w))\}$

5.1 then STOP ($L(G) \notin \mathcal{Q}$);

6. If $\upsilon(L(G)) = \upsilon(L(G) - \{w\}) > \upsilon(L(G) - N_{L(G)}(w))$

6.1 then set $L(G) = L(G) - \{w\}$ and goto step 1;

7. If $\upsilon(L(G)) = \upsilon(L(G) - N_{L(G)}(w)) > \upsilon(L(G) - \{w\})$

7.1 then set $L(G) = L(G) - N_{L(G)}(w)$ and goto step 1;

8. If the optimal solution of $(P_{L(G)-\{w\}})$ or $(P_{L(G)-N_{L(G)}(w)})$ is integer

8.1 then STOP ($L(G) \in \mathcal{Q}$);

8.2 else if $W \neq \emptyset$ then goto step 4;

9. Choose $v \in V(L(G))$ and

9.1 apply the algorithm to the graph $L(G) - \{v\}$ and if

$$L(G) - \{v\} \in \mathcal{Q}$$

then STOP ($L(G) \in \mathcal{Q}$) else goto 9.2;

9.2 apply the algorithm to the graph $L(G) - N_{L(G)}(v)$ and if

$$L(G) - N_{L(G)}(v) \in \mathcal{Q}$$

then STOP ($L(G) \in \mathcal{Q}$) else STOP ($L(G) \notin \mathcal{Q}$);

End.

The main steps of the algorithm are consequence of the following results.

The step 2 is obvious. The step 3 follows from the fact that when $\lambda_{\min}(A_{L(G)}) \notin \mathbb{Z}$, then $(P_{L(G)})$ has an optimal solution, x^*, which is unique, and hence $L(G) \in \mathcal{Q}$ if and only if $x^* \in \{0, 1\}^{|V(L(G))|}$. The step 5 follows from Theorem 4. The steps 6 and 7 are direct consequence of Theorem 5 and so, once obtained a subgraph of $L(G)$, $L(G')$, for which it is possible to get a conclusion, then $L(G) \in \mathcal{Q}$ if and only if $L(G') \in \mathcal{Q}$. The step 8 follows taking into acount that if $\upsilon(L(G)) = \upsilon(L(G) - U)$ then the optimal solutions of $(P_{L(G)-U})$ define optimal solutions for $(P_{L(G)})$. Finally, the step 9 follows from theorem 6 implying the recursive execution of the algorithm.

The step 9 is reached when $\forall v \in V(L(G))$

$$\begin{aligned} \upsilon(L(G)) &= \upsilon(L(G) - \{v\}) \\ &= \upsilon(L(G) - N_{L(G)}(v)) \end{aligned}$$

and

$$\begin{aligned} \lambda_{min}(A_{L(G)}) &= \lambda_{min}(A_{L(G)-\{v\}}) \\ &= \lambda_{min}(A_{L(G)-N_{L(G)}(v)}) \end{aligned}$$

and, in such case, a branching strategy it is performed in order to know if $L(G) \in \mathcal{Q}$ or not.

4. Numerical Example

It follows a numerical example for the recognition of a graph with a perfect matching.

Let us consider the graph, G, and the corresponding graph $L(G)$ both depicted in Figure 1. Then $\lambda_{\min}(A_{L(G)}) = -2$.

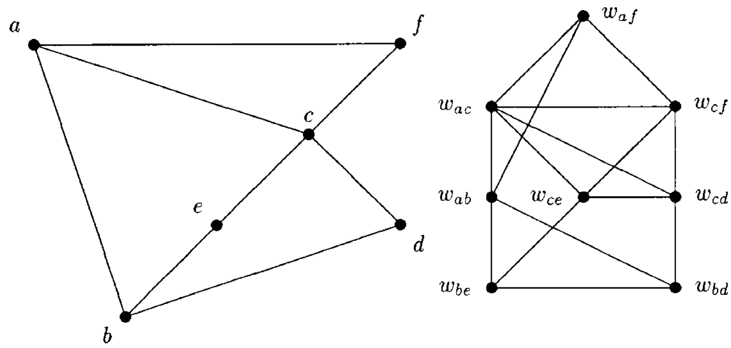

Figure 1. Graphs G and $L(G)$.

Since G is connected, in order to know if $G \in \mathcal{G}$, we may determine a non negative solution of the system $(\frac{1}{2}A_{L(G)} + I_8)x = \hat{e}$, if such solution exists (otherwise, according to theorem 8 $L(G) \notin \mathcal{Q}$), that is, we may try to find a non negative solution for the system

$$\begin{array}{c} \\ w_{af} \\ w_{ac} \\ w_{cf} \\ w_{ab} \\ w_{ce} \\ w_{cd} \\ w_{be} \\ w_{bd} \end{array} \begin{array}{c} \begin{array}{cccccccc} w_{af} & w_{ac} & w_{cf} & w_{ab} & w_{ce} & w_{cd} & w_{be} & w_{bd} \end{array} \\ \left(\begin{array}{cccccccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{array} \right) \end{array} \begin{bmatrix} x_{af} \\ x_{ac} \\ x_{cf} \\ x_{ab} \\ x_{ce} \\ x_{cd} \\ x_{be} \\ x_{bd} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

As we get $x^*_{L(G)}$, with components $x^*_{af} = 1$, $x^*_{ac} = x^*_{cf} = x^*_{ab} = 0$ and $x^*_{ce} = x^*_{cd} = x^*_{be} = x^*_{bd} = \frac{1}{2}$, which is an optimal solution for $(P_{L(G)})$, then $\upsilon(L(G)) = 3 = \frac{|V(G)|}{2}$ and hence $G \in \mathcal{G}$. Therefore the algorithm may be applied.

Since $x^*_{L(G)} \notin \{0, 1\}^8$, by step 2 we proceed with step 3. By step 3, since $\lambda_{min}(A_{L(G)}) = -2$, we proceed with step 4. Choosing, for instance, the node in $W = V(L(G))$ with maximum degree in $L(G)$, w_{ac}, the graphs $L(G) - \{w_{ac}\}$ and $L(G) - N_{L(G)}(w_{ac})$, depicted in Figure 2, are obtained. Since $x^*_{L(G)-\{w_{ac}\}}$, with components $x^*_{af} = 1$, $x^*_{cf} = x^*_{ab} = 0$ and $x^*_{ce} = x^*_{cd} = x^*_{be} = x^*_{bd} = \frac{1}{2}$, and $x^*_{L(G)-N_{L(G)}(w_{ac})}$, with components $x^*_{bd} = 0$ and $x^*_{ac} = x^*_{be} = 1$ are optimal solutions for $(P_{L(G)-\{w_{ac}\}})$ and $(P_{L(G)-N_{L(G)}(w_{ac})})$, respectively, we have

$$3 = \upsilon(L(G)) = \upsilon(L(G) - \{w_{ac}\}) > \upsilon(L(G) - N_{L(G)}(w_{ac})) = 2.$$

Then, applying steps 5 and 6, we return to steps 1, 2, 3 and 4 with the line graph $L(G) - \{w_{ac}\}$. Choosing, from W, the node w_{ce} we get the graphs depicted in Figure 3.

Now, since $x^*_{L(G)-\{w_{ac},w_{ce}\}}$, with components $x^*_{af} = x^*_{cd} = x^*_{be} = 1$ and $x^*_{cf} = x^*_{ab} = x^*_{bd} = 0$, and $x^*_{L(G)-\{w_{ac}\}-N_{L(G)-\{w_{ac}\}}(w_{ce})}$, with components $x^*_{af} = x^*_{ce} = x^*_{bd} =$

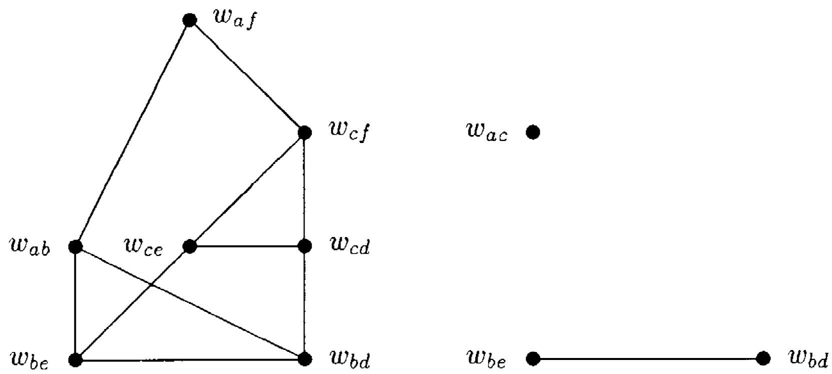

Figure 2. Graphs $L(G) - \{w_{ac}\}$ and $L(G) - N_{L(G)}(w_{ac})$.

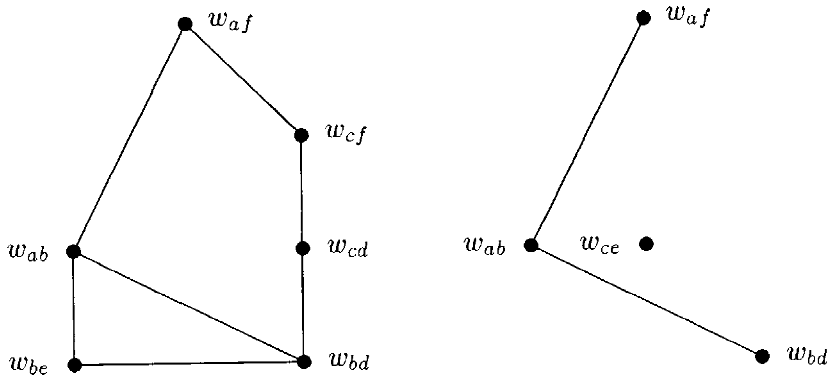

Figure 3. Graphs $L(G) - \{w_{ac}, w_{ce}\}$ and $L(G) - \{w_{ac}\} - N_{L(G)-\{w_{ac}\}}(w_{ce})$.

1 and $x^*_{ab} = 0$, are optimal solutions for $(P_{L(G)-\{w_{ac},w_{ce}\}})$ and $(P_{L(G)-\{w_{ac}\}-N_{L(G)-\{w_{ac}\}}(w_{ce})})$, respectively, then

$$\begin{aligned} \upsilon(L(G) - \{w_{ac}\}) &= \upsilon(L(G) - \{w_{ac}, w_{ce}\}) \\ &= \upsilon(L(G) - \{w_{ac}\} - N_{L(G)-\{w_{ac}\}}(w_{ce})), \end{aligned}$$

and hence, reaching to step 8, since the optimal solutions of $(P_{L(G)-\{w_{ac},w_{ce}\}})$ and $(P_{L(G)-\{w_{ac}\}-N_{L(G)-\{w_{ac}\}}(w_{ce})})$ are both integer solutions we conclude that $L(G) \in \mathcal{Q}$ and the algorithm stops.

It must be noted that since the components of the optimal the solutions for $(P_{L(G)-\{w_{ac},w_{ce}\}})$ and $(P_{L(G)-\{w_{ac}\}-N_{L(G)-\{w_{ac}\}}(w_{ce})})$, are 0–1 and $\upsilon(L(G)) = \upsilon(L(G) - \{w_{ac}, w_{ce}\}) = \upsilon(L(G) - \{w_{ac}\} - N_{L(G)-\{w_{ac}\}}(w_{ce}))$, both define characteristic vectors of maximum stable sets of $L(G)$ and then the sets of edges

$$M_1 = \{\{a, f\}, \{c, d\}, \{b, e\}\} \quad \text{and} \quad M_2 = \{\{a, f\}, \{c, e\}, \{b, d\}\}$$

are perfect matchings for G.

5. Extensions and Related Results

The algorithm for recognising a graph with a perfect matching can be easily extended to the determination of maximum matchings of arbitrary graphs. In fact, assuming that G is connected and has at least one edge, in order to determine a maximum matching of G, M^*, we can apply the following algorithm, where it is assumed that W_k is disjoint from $V(G_k)$.

0. *Algorithm* (to find a maximum matching of G)

1. Set $k = 0$ and $G_0 = G$;

2. If $L(G_k) \in \mathcal{Q}$ then goto step 7;

3. If $|V(G_k)|$ is odd

3.1 then set $W_k = \{w_k\}$;
3.2 else set $W_k = \{w_k^1, w_k^2\}$;

4. Set $V(G_{k+1}) = V(G_k) \cup W_k$;

5. Set $E(G_{k+1}) = E(G_k) \cup \{\{v, w\} : v \in V(G_k), w \in W_k\}$;

6. Set $k = k + 1$ and goto step 2;

7. If M is a perfect matching for G_k then set $M^* = E(M) \cap E(G)$.

End.

It must be noted that in worst case the algorithm ends when $|V(G)| - 2$ nodes are added to G. In fact, assuming that G is connected and has at least one edge, according to the above procedure, if there is k such that $|V(G_k)| = 2|V(G)| - 2$ then $L(G_k) \in \mathcal{Q}$.

Since according to Las Vergnas (1975) every connected claw-free graph of even order has a perfect matching, we may conclude that every line graph of a connected graph with even size has a perfect matching, and therefore we have the following corollary of Theorem 7.

COROLLARY 9. *If G is a connected graph such that $|E(G)|$ is even then $L(L(G)) \in \mathcal{Q}$.*

Proof. Since every line graph is claw-free and G is connected of even size, then $L(G)$ is connected of even order. Therefore, according to Las Vergnas (1975) $L(G)$ has a perfect matching and then, by Theorem 7, $L(L(G)) \in \mathcal{Q}$, since also $L(L(G))$ is connected. □

As an immediate consequence of the result of Las Vergnas (1975), if G is connected and has an even number of edges, then $E(G)$ can be partitioned in paths of length 2.

An edge $e \in E(G)$ is called α–critical if $\alpha(G - \{e\}) > \alpha(G)$, where $G - \{e\}$ denotes the graph obtained from G such that $V(G - \{e\}) = V(G)$ and $E(G - \{e\}) = E(G) \setminus \{e\}$. Then, as a direct consequence of Theorem 7, we may conclude that a connected graph G of size $|E(G)| \geqslant 2$ has a perfect matching if and only if

$L(G)$ has no α–critical edges. Another consequence of Theorem 7 is that if G is an Hamiltonian graph of even order, then $L(G) \in \mathcal{Q}$.

Finally, since the inequalities (1.5) are fulfilled in equality form for graphs in the class $\mathcal{Q}$ using optimal integer solutions of $(P_G^{\phi^*})$ (and then also of (P_G)) we may conclude that for these graphs spurious solutions to the Motzkin-Strauss program (1.2) (i.e., those which only deliver the stability number via the optimal objective value but do not serve to retrieve a maximum stable set) are impossible.

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